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Note

A Euclidean Ramsey theorem

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Abstract

In this note we shall prove a geometric Ramsey theorem. Let T be a triangle with angles 30, 60 and 90 degrees, and with hypotenus of unit length. Then the theorem says that if one three-colors the 3-space, then there is always a copy of T with monochromatic vertices. We shall also show that there is a 12-coloration of the space in which there is no copy of T with monochromatic vertices.

1. Proof of the theorem

Suppose that there is no copy of T with monochromatic vertices.

Consider a line segment GH of length $\sqrt{3}/2$ whose vertices are the same color, say green. Let g and h be two circles of radius $1/2$ whose centres are G and H respectively and whose planes are perpendicular to GH. Then all points of g and h must be not green, say red or blue. See Fig. 1.

Consider a regular hexagon whose circumcircle is g. Then it is clear that there is only two possibilities to color its vertices. See Fig. 2.

From now on we shall consider regular hexagonal prisms with hexagonal faces inscribed to g and h and rectangular faces parallel to GH.

We have to consider different cases according to the type of the coloring of the two hexagonal faces of our prism.

Case 1.a: If both of them are of type B and there is a blue vertex above an other one then we are done. See Fig. 3.

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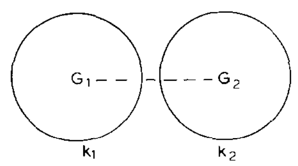

Fig. 1.

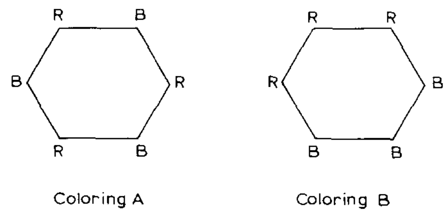

Fig. 2.

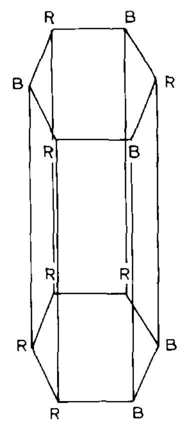

Fig. 3.

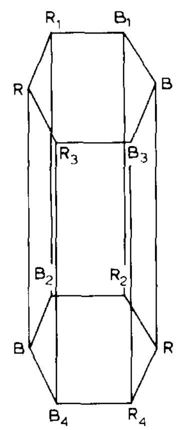

Fig. 4.

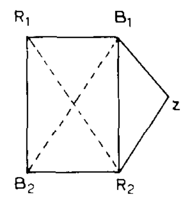

Fig. 5.

Case 1.b: If both of the hexagonal faces are of type B but there is no blue vertex above an other one than we consider the centres X and Y of the rectangles $R1B1R2B2$ and $R3B3R4B4$. See Fig. 4.

The segments $B1B3$ and $R1R3$ force X and Y to be green. Now consider the regular hexagon shown in Fig. 5.

It is easy to see that if Z is red then $ZR1R2$, if Z is blue then $ZB1B2$ and if Z is green then ZXY is a monochromatic copy of T.

Case 2: If one hexagonal face is of type A and the other one is of type B, then we always have a blue vertex above an other blue one, and we are ready as in the Case 1.a. See Fig. 3.

Case 3.a: If both hexagonal faces are of type A, and there are blue vertices above red ones then we can consider the center of two opposite hexagons (they must be green) and complete the proof in the same way as in Case 1.b. See Fig. 6.

Case 3.b: Both hexagonal faces are of type A but there are red vertices above red ones.

This is the most complicated case. Consider centers of the rectangular faces. All they are green. (As in previous cases.)

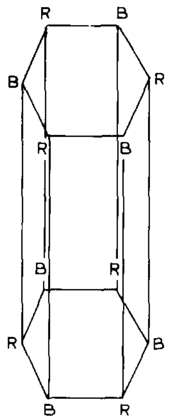

Fig. 6.

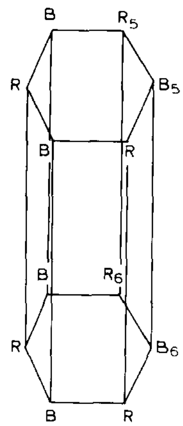

Fig. 7.

Let us rotate our prism around GH. If there exists a rotated copy of it whose coloring is not of Case 3.b then we are done since we have already shown a monochromatic copy of T in any other cases. Therefore, the circle f determined by rotated images of face centers must be entirely green. Observe that f is of diameter $\sqrt{3}/2$, that is why all pairs of opposite points of f can play the role of G and H. Because of our contrary hypothesis, for any such choice of G and H Case 3.b must occur, which proves that all the Thales-sphere of GH is green.

But it also proves (by our contrary hypothesis), that for any line segment EF of length $\sqrt{3}/2$ whose vertices are monochromatic, the Thales-sphere of EF is monochromatic, E and F included.

But it is a contradiction, because we can see at Fig. 7 that Thales spheres of R_5R_6 and B_5B_6 intersect.

We have proven that there always exists a copy of T with monochromatic vertices.

2. An example of a 12-coloring of the 3-space in which there is no copy of T with monochromatic vertices

Denote the different colors by natural numbers $1, 2, 3, \ldots, 12$. The color of any point (x, y, z) will be the same as the color of the point $(x, y, 0)$. The coloring of the plane $z = 0$ is shown in Fig. 8.

So we color a regular hexagonal lattice of side length $\sqrt{3}/8$. Each hexagon is entirely monochromatic and hexagons whose distance is $\sqrt{3}/2$ have the same color. Points at the border of two hexagons belong to the hexagon whose centre is lexicographically larger. (As an ordered pair (x, y).)

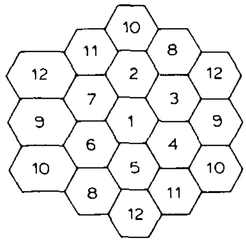

Fig. 8.

One can easily check that this coloring does not contain any copy of T with monochromatic vertices. (One hexagon is too small for this purpose and two hexagons with same color have too large distance.)